

# Magnetosonic solitons in a fermionic quantum plasma

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(Received 10 April 2007; revised manuscript received 27 September 2007; published 19 December 2007)

Starting from the governing equations for a quantum magnetoplasma including the quantum Bohm potential and electron spin-1/2 effects, we show that the system of quantum magnetohydrodynamic (QMHD) equations admits rarefactive solitons due to the balance between nonlinearities and quantum diffraction and tunneling effects. It is found that the electron spin-1/2 effect introduces a pressurelike term with negative sign in the QMHD equations, which modifies the shape of the solitary magnetosonic waves and makes them wider and shallower. Numerical simulations of the time-dependent system shows the development of rarefactive QMHD solitary waves that are modified by the spin effects.

 DOI: [10.1103/PhysRevE.76.067401](https://doi.org/10.1103/PhysRevE.76.067401)

PACS number(s): 52.27.Gr, 67.57.Lm

## I. INTRODUCTION

There is currently a great deal of interest in collective quantum effects in plasmas [1–12]; many of these studies are motivated by recent experimental progress and techniques [13–16] and also by possible astrophysical applications [12,17–20]. In particular, magnetohydrodynamic (MHD) plasmas are of interest in such astrophysical applications. However, in strong magnetic fields, single-electron effects that depend on the electron spin properties, such as Landau quantization, will be important. It is thus not surprising that collective spin effects can influence the wave propagation in a strongly magnetized quantum plasma [12,21–23]. Moreover, the recent progress in producing ultracold plasmas in terms of Rydberg states [24,25] may offer an interesting experimental environment for quantum plasma dynamics. In such cold plasmas, the thermal energy of the particles can be very small compared to the Zeeman energy of the particles in magnetic fields. Thus, collective spin properties of quantum plasmas may be possible to detect in the near future.

In this Brief Report, we will show that the balance between the nonlinear plasma and quantum effects gives rise to magnetosonic solitons. Using the governing equations for quantum MHD (QMHD) plasmas with tunneling and spin effects included, we derive a Sagdeev potential for the one-dimensional system. We show that, in a magnetized quantum plasma, the electron spin-1/2 effect can strongly modify the amplitude and width of rarefactive solitons.

## II. GOVERNING EQUATIONS

We begin by presenting the general governing equations for a quantum magnetoplasma in which the electron spin-1/2 effects are included. We define the total mass density  $\rho \equiv (m_e n_e + m_i n_i)$ , the center-of-mass fluid flow velocity  $\mathbf{V} \equiv (m_e n_e \mathbf{v}_e + m_i n_i \mathbf{v}_i) / \rho$ , and the current density  $\mathbf{j} = -en_e \mathbf{v}_e + en_i \mathbf{v}_i$ . Here  $m_e$  ( $m_i$ ) is the electron (ion) mass,  $n_e$  ( $n_i$ ) is the electron (ion) number density,  $\mathbf{v}_e$  ( $\mathbf{v}_i$ ) is the electron (ion) fluid velocity, and  $e$  is the magnitude of the electron charge. From the general set of spin-fluid equations [12], the corresponding QMHD equations can be derived [23]. From these, we immediately obtain the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0. \quad (1)$$

Assuming quasineutrality, i.e.,  $n_e \approx n_i$ , the momentum conservation equation reads

$$\rho \left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \mathbf{j} \times \mathbf{B} - \nabla P + \mathbf{F}_Q, \quad (2)$$

where  $P$  is the scalar pressure in the center-of-mass frame, the current is given by  $\mathbf{j} = \mu_0^{-1} \nabla \times (\mathbf{B} - \mu_0 \mathbf{M})$ ,  $\mathbf{M} = (\mu_B \rho / m_i) \tanh(\mu_B B / k_B T_e) \hat{\mathbf{B}}$  is the plasma magnetization due to the electron spin, and [12,23]

$$\mathbf{F}_Q = \frac{\hbar^2 \rho}{2m_e m_i} \nabla \left( \frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} \right) + \frac{\mu_B \rho}{m_i} \tanh \left( \frac{\mu_B B}{k_B T_e} \right) \nabla B \quad (3)$$

is the quantum force due to collective tunneling and spin alignment. Here  $\mu_B = e\hbar / 2m_e$  is the magnitude of the Bohr magneton,  $\hbar$  is the Planck constant divided by  $2\pi$ , and  $c$  is the speed of light in vacuum. The generalized Faraday law takes the form

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left\{ \mathbf{V} \times \mathbf{B} - \frac{[\nabla \times (\mathbf{B} - \mu_0 \mathbf{M})] \times \mathbf{B}}{en_e \mu_0} - \eta \mathbf{j} - \frac{m_e}{e^2 \mu_0} \left[ \frac{\partial}{\partial t} - \left( \frac{\nabla \times \mathbf{B}}{e \mu_0 n_e} \right) \cdot \nabla \right] \frac{\nabla \times \mathbf{B}}{n_e} - \frac{\mathbf{F}_Q}{en_e} \right\}, \quad (4)$$

where  $\eta$  is the plasma resistivity.

## III. SPIN SOLITONS

Next, we assume that the magnetic field is along the  $z$  direction such that  $\mathbf{B} = B(x, t) \hat{\mathbf{z}}$ , while we have the velocity  $\mathbf{V} = V(x, t) \hat{\mathbf{x}}$  and the density  $\rho(x, t)$ . With this, the governing equations reduce to

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho V) = 0, \quad (5)$$

$$\begin{aligned} \frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} = & -\frac{B}{\mu_0 \rho} \frac{\partial B}{\partial x} - C_s^2 \frac{\partial}{\partial x} \ln \rho \\ & + 2c^2 \lambda_C^2 \frac{m_e}{m_i} \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} \right) \\ & + \frac{\mu_B}{m_i \rho} \frac{\partial}{\partial x} \left[ \rho B \tanh \left( \frac{\mu_B B}{k_B T_e} \right) \right], \end{aligned} \quad (6)$$

and

$$\frac{\partial B}{\partial t} + \frac{\partial}{\partial x} (BV) - \lambda \frac{\partial^2 B}{\partial x^2} = 0. \quad (7)$$

Here  $\lambda_C = c/\omega_C = \hbar/2m_e c$  is the Compton wavelength,  $\omega_C$  is the Compton frequency,  $C_s = [k_B(T_e + T_i)/m_i]^{1/2}$  is the sound speed,  $\lambda = \eta/\mu_0$  is the magnetic diffusivity, the last term in Eq. (6) is the spin force divided by  $m_i$ , and we have neglected the inertial term in the Faraday law (7).

If the resistivity is weak, we may neglect the last term in the Faraday law (7), and obtain the frozen-in-field condition  $\rho = \rho_0 b$ , where  $b = B/B_0$ , with the background values denoted by the zero index. Then, Eqs. (6) and (7) form a closed system, taking the form

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial}{\partial x} \left( \frac{V^2}{2} \right) = & -C_A^2 \frac{\partial b}{\partial x} - C_s^2 \frac{\partial}{\partial x} \ln b \\ & + 2c^2 \lambda_C^2 \frac{m_e}{m_i} \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{b}} \frac{\partial^2 \sqrt{b}}{\partial x^2} \right) \\ & + \frac{k_B T_e}{m_i} \frac{\partial}{\partial x} \{ \ln[\cosh(\varepsilon b)] + \varepsilon b \tanh(\varepsilon b) \} \end{aligned} \quad (8)$$

and

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial x} (bV) = 0, \quad (9)$$

where we have introduced the Alfvén speed  $C_A = (B_0^2/\mu_0 \rho_0)^{1/2}$  and the temperature-normalized Zeeman energy  $\varepsilon = \mu_B B_0/k_B T_e$ .

We now normalize our variables as  $\bar{t} = \omega_{ci} t$ ,  $\bar{x} = (\omega_{ci}/C_A)x = (\omega_{pi}/c)x$  [where  $\omega_{pi} = (n_0 e^2/\epsilon_0 m_i)^{1/2}$  is the ion plasma frequency],  $v = V/C_A$ , and  $c_s = C_s/C_A$ . We then obtain

$$\begin{aligned} \frac{\partial v}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}} \left( \frac{v^2}{2} \right) = & -\frac{\partial b}{\partial \bar{x}} - c_s^2 \frac{\partial}{\partial \bar{x}} \ln b + 2 \frac{\omega_{pe}^2}{|\omega_{ce}| \omega_C} \frac{\partial}{\partial \bar{x}} \left( \frac{1}{\sqrt{b}} \frac{\partial^2 \sqrt{b}}{\partial \bar{x}^2} \right) \\ & + v_B^2 \frac{\partial}{\partial \bar{x}} \{ \ln[\cosh(\varepsilon b)] + \varepsilon b \tanh(\varepsilon b) \}, \end{aligned} \quad (10)$$

with  $v_B^2 = k_B T_e/m_i C_A^2 = (1/\varepsilon)(\mu_B B_0/m_i C_A^2)$  and

$$\frac{\partial b}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}} (bv) = 0, \quad (11)$$

where for simplicity we drop the overbars on the normalized coordinates.

Next, we assume that  $v$  and  $b$  are functions of  $\xi = x - v_0 t$ , where  $v_0$  is a constant speed (normalized by  $C_A$ ). Then Eq.

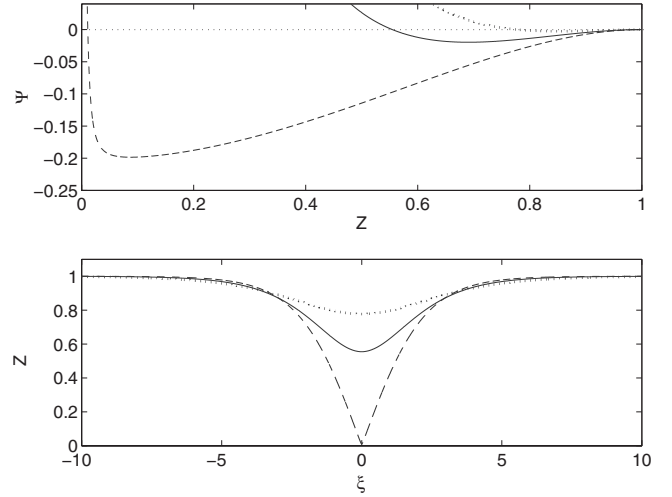


FIG. 1. Sagdeev potential  $\Psi(Z)$  (upper panel) and the profile of the solitary wave  $Z(\xi)$  (lower panel), for  $v_0=0.01$  (dashed lines), 0.5 (solid lines), and 0.7 (dotted lines). The other parameters are  $\varepsilon=5$ ,  $c_s=0.1$ ,  $v_B=0.2$ , and  $|\omega_{ce}| \omega_C/\omega_{pe}^2=1$ .

(11) can be integrated as  $v = v_0(1 - 1/b)$ , where we used the boundary conditions  $b=1$  and  $v=0$  at  $|\xi| = \infty$ , and Eq. (10) can be integrated twice to obtain

$$\left( \frac{dZ}{d\xi} \right)^2 + \Psi(Z) = 0, \quad (12)$$

where  $Z = \sqrt{b}$ , and the Sagdeev potential [26] for our purposes reads

$$\begin{aligned} \Psi = & \frac{|\omega_{ce}| \omega_C}{\omega_{pe}^2} \left\{ \frac{v_0^2}{4} \left( Z - \frac{1}{Z} \right)^2 - \frac{1}{4} (Z^2 - 1)^2 \right. \\ & - \frac{c_s^2}{2} [Z^2 \ln(Z^2) - Z^2 + 1] \\ & \left. + \frac{v_B^2}{4} \left[ Z^2 \ln \left( \frac{\cosh(\varepsilon Z^2)}{\cosh(\varepsilon)} \right) - \varepsilon \tanh(\varepsilon) (Z^2 - 1) \right] \right\}. \end{aligned} \quad (13)$$

In deriving (13) we have used the condition  $\Psi(1)=0$ . In Figs. 1 and 2, we have plotted the Sagdeev potential as well as the profiles of the corresponding solitary waves for different sets of parameters. The solitary waves have only sub-Alfvénic speeds and are characterized by a localized depletion of the magnetic field and density. In Fig. 1, we see that the solitary waves increase their amplitudes for smaller speeds. In the limit of zero speed, we have rarefactive solitons with a zero density at its center. The influence of the electron spin-1/2 effect on the solitary waves is displayed in Fig. 2, where we see that larger values of  $\varepsilon$  lead to wider solitary waves with shallower density and magnetic field depletions. In order to study the influence of the spin pressure on the nonlinear dynamics of our system, we have solved the time-dependent system of equations (10) and (11) for different values of the spin pressure parameter  $\varepsilon$ . As an initial condition at  $t=0$ , we took a magnetic field with a local depletion in the form Gaussian pulse

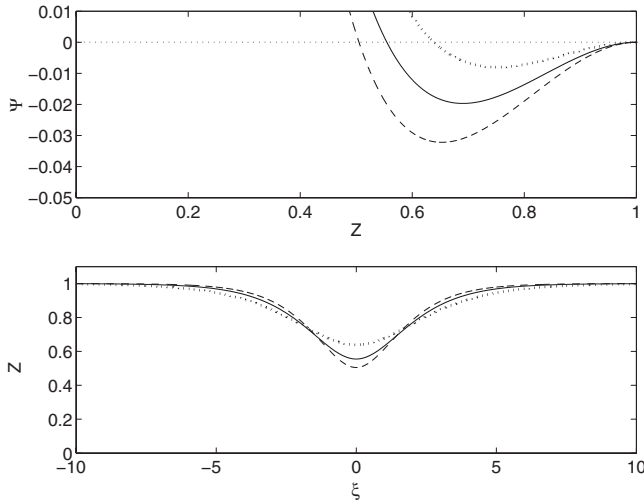


FIG. 2. Sagdeev potential  $\Psi(Z)$  (upper panel) and the profile of the solitary wave  $Z(\xi)$  (lower panel), for  $\varepsilon=1$  (dashed lines), 5 (solid lines), and 10 (dotted lines). The other parameters are  $v_0=0.5$ ,  $c_s=0.1$ ,  $v_B=0.2$ , and  $|\omega_{ce}|\omega_C/\omega_{pe}^2=1$ .

$b=1-0.5\exp(-x^2/100)$ , while the velocity  $v$  was set to zero. For  $\varepsilon=5$ , we see in the left-hand column of panels in Fig. 3 that the initial pulse develops into two counterpropagating pairs of rarefactive solitary waves, where the smaller pulse in the pair propagates with a somewhat larger speed,  $\sim 0.75C_A$ , than the larger one, which propagates with a speed of  $\sim 0.65C_A$ . For a larger value  $\varepsilon=10$ , displayed in the right-hand panels of Fig. 3, the pulse develops into two counterpropagating pulses that propagate with somewhat lower speed,  $\sim 0.4C_A$ , and they are wider and of smaller amplitude than the large-amplitude pulses for  $\varepsilon=5$ . All pulses are rarefactive and are propagating with sub-Alfvénic speed, in agreement with our analysis in Figs. 1 and 2.

#### IV. SUMMARY AND DISCUSSION

In the numerical examples of the previous section, the normalized Zeeman energy  $\varepsilon$  played a crucial role. In particular, the spin contribution to the soliton dynamics is enhanced when the Zeeman energy is of the order of or greater than 1 (we note, however, that other parameters play a role in forming the necessary shape of the Sagdeev potential). Thus, it is natural to investigate what type of parameter values correspond to  $\varepsilon \geq 1$ . For astrophysical plasmas, such as in pulsar magnetospheres, we can have  $B_0 \leq 10^{10}$  T [20], implying that  $\varepsilon \geq 1$  for  $T_e \leq 10^9$  K, i.e., not a very severe constraint. However, in such environments, the plasma often has relativistic temperatures and flows, and a relativistic formal-

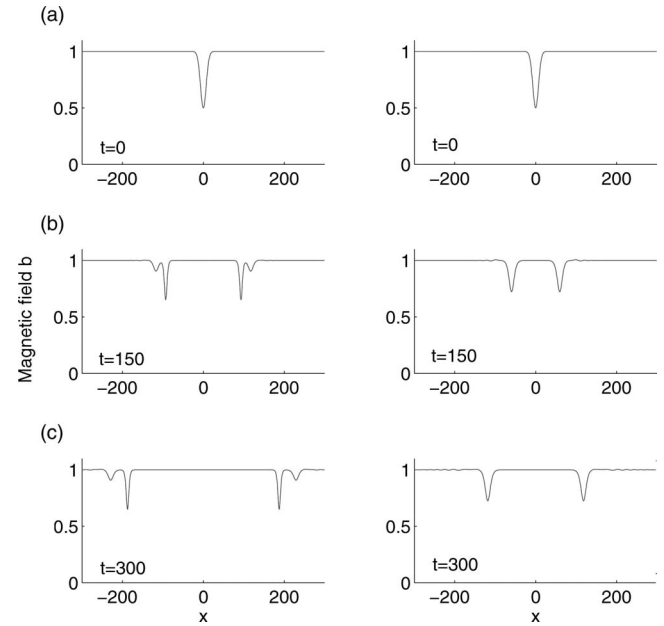


FIG. 3. Time-dependent dynamics of the normalized magnetic field  $b$ , for  $\varepsilon=5$  (left column) and 10 (right column). The other parameters are  $c_s=0.1$ ,  $v_B=0.2$ , and  $|\omega_{ce}|\omega_C/\omega_{pe}^2=1$ .

ism should be used. In the case of Rydberg plasmas [24,25], where the temperature can go as low as millikelvins, we see that the Zeeman energy is greater than 1 for external magnetic field  $B_0 \geq 10^{-3}$  T. Thus, in such ultracold laboratory systems, a very weak external magnetic field would make spin effects important for the formation of solitons, and the theory presented here could therefore be checked experimentally.

In conclusion, we have investigated the effects of the quantum Bohm potential and the electron spin 1/2 on the existence of magnetosonic solitary waves in a magnetized quantum plasma. The solitary waves exist due to a balance between the nonlinearities and the dispersion induced by the electron quantum diffraction and tunneling effects associated with the quantum Bohm potential. The spin introduces an additional negative pressurelike term in the quantum momentum equation, with the effect that solitary waves become wider and have shallower density depletions for larger values of the Zeeman energy  $\varepsilon = \mu_B B / k_B T_e$ . We note that the spin term in the Sagdeev potential (13) can dominate the dynamics in the regime of  $C_s^2, C_A^2 \ll C_A^2 v_B^2 \varepsilon$ . This regime corresponds to a dense quantum plasma with an ambient magnetic field, such that  $\omega_{ce}\omega_C \ll \omega_{pe}$  and  $k_B(T_e + T_i) \ll \mu_B B_0$ . Thus, the spin of the electrons collectively modifies the quantum dynamics of the MHD plasma significantly.

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